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Five-Dimensional Space-Time: Mass and the Fundamental Length

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Abstract

Relativistic physics is described by a sixteen-component hypercomplex number system which reduces to the eight-component complex quaternion system when rest mass → 0. Rest mass is identified with a scale setting, cyclic, fifth dimension. The Lorentz group is generalized to rotations in five dimensions, a ten-parameter group. Special Relativity and General Relativity are tentatively welded into one unified covariance scheme in five-dimensional space-time.

1. *Introduction*

What we really know about relativistic quantum physics can be summarized by writing the Dirac equation coupled to the Maxwell equation. The spin-$\frac{1}{2}$ particle mass is put in as an *ad hoc* parameter of unknown origin. Through second quantization, we add commutator and anticommutator postulates to give the Pauli exclusion principle, which is also of unknown origin.

We describe the space-time movement of energy lumps using three-dimensional space, but no one knows why. There are so many other unanswered, basic questions about the subnuclear world, that it is obvious we are stumbling blindly into a whole new world of ideas and concepts. Our old ways will need considerable generalization to describe it. My plea is for very open minds. Wild innovations are needed. Some subset of these will open the way to the subnuclear world. Here we present another beautiful but very speculative view of relativistic quantum theory. It encompasses what we think is valid but puts it in a radical conceptual framework. I think it is a serious mistake to ask fundamental questions about quantum relativity while ignoring the requirements of General Relativity—covariance under 'any' curvilinear coordinate change and also nonstatic curved space-time. For too long this has been the practice of particle theorists.

The sixteen-component extension of the complex quaternion number system provides a mathematical formalism where General Relativity and Quantum Mechanics can be unified (Edmonds, 1974a, b, c). This number system is isomorphic to $\{e_0, ie_k\} \otimes \{e_0, ie_1, e_2, e_3\}$ where e_μ and ie_μ are the basis elements of the complex quaternions. For a world without rest mass, only the complex quaternions are needed. Mass brings to nature a scale of length as well as the more complicated number system.

In this paper we develop a five-dimensional space-time formalism in which mass is associated with the fifth dimension. This dimension is assumed subatomic or cyclic. Its periodic length gives the scale of length to nature. Periodic coordinates are quite familiar to solid state physics. They give Bloch waves instead of plane waves for quasiparticles moving through the crystal lattice. We assume that t and χ^k are not periodic except possibly on a very large scale, should the universe be closed due to its curvature.

Previously we have considered 5- and 8-vectors and associated the fifth dimension with time in the cosmic rest frame, However, the natural generalization of the Lorentz group seems to suggest rotational symmetry in five dimensions. Also mass is directly associated with the fifth dimension and the fundamental length. Therefore, a microscopic, periodic fifth dimension seems more reasonable at present.

2. *Space-Time Structure*

Space-time is postulated to have the following hypercomplex number structure:

$$\begin{aligned} &dx = dx^\mu b_\mu, \qquad \mu = 0, 1, 2, 3, 4; \qquad b_0 \to (e_0), \qquad b_k \to (e_k), \\ &k = 1, 2, 3, \qquad b_4 \to (if_0) \text{ in flat space limit}; \qquad \chi^4 + l \equiv \chi^4; \\ &dx^0 = c\, dt \end{aligned} \tag{2.1}$$

Here l is the fundamental length. In practice $b_\mu = b_\mu(x)$, because the universe is curved, both macroscopically and near all energy concentrations such as macroscopic bodies or fundamental particles. Under a general curvilinear coordinate change $dx^\mu b_\mu$ is invariant. The metric is defined as

$$g_{\mu\nu} \equiv \tfrac{1}{2}[(b_\mu{}^\wedge b_\nu) + (\ )^\wedge] = g_{\nu\mu} \propto e_0 \tag{2.2}$$

The space curvature is determined from Einstein's gravity equation, *generalized* to five dimensions. This suggests something like

$$b_\mu{}^\wedge(D^\mu D^\nu - D^\nu D^\mu) b_\nu = b_\mu{}^\wedge M^{\mu\nu} b_\nu, \qquad D^\mu(g_{\nu\lambda}) \equiv 0 \tag{2.3}$$

Notice that $b_\mu D^\nu \neq D^\nu b_\mu$ in *curved* space. This has subtle but important consequences in formulating wave equations. Many forms which are equivalent in flat space are not when b_μ doesn't commute with the covariant derivative D^ν. For curvilinear coordinates in flat space they can always commute, which shows that accelerated frames are *not* equivalent to a gravitational field. Equation (2.3) is a gravity field wave equation. The field can be written

$$b_\mu(x) = b_\mu{}^0(x)(e_0) + b_\mu{}^k(x)(e_k) + b_\mu{}^4(x)(if_0), \qquad \mu = 0, 1, 2, 3, 4 \tag{2.4}$$

I think, but this needs to be proven. In the flat, cartesian limit $b_\mu{}^\nu \to \delta_\mu{}^\nu$. The wave equation discussion to follow can be thought to represent such a curved five-dimensional space, though we shall use the flat space notation. Should equation (2.3) turn out to require that adding (if_0) to space-time requires adding the other eleven coordinates $\{ie_0, ie_k, f_0, f_k, if_k\}$ or at least (f_0), then space-time is probably much more complicated than the simple five-dimensional formulation considered here.

3. *Wave Equation Structure*

All wave equations, like equation (2.3), are form invariant for any invertible coordinate transformation (Einstein covariance). Nature appears to have *additional* restrictions on its wave equations. The physical reason in curved space is unclear; maybe there is not any. We *postulate* that free particle wave equations are form invariant under

$$b_\mu \to b'_\mu = \mathscr{L}^\dagger b_\mu \mathscr{L}, \qquad \mathscr{L}\mathscr{L}^\wedge \equiv 1e_0 \tag{3.1}$$

For the special case $\mu = 0, 1, 2, 3$ this gives the Lorentz group. In five or higher dimensional spaces, we find that $\mathscr{L}\mathscr{L}^\wedge$ is a ten parameter Lie group with generators $\{ie_k, e_k, if_0, if_k\}$. Since $\mathscr{L}$ is a sixteen-component hypercomplex number we must verify that b'_μ is still a 5-vector. It turns out that under $\mathscr{L}^\dagger(\,)\,\mathscr{L}$ we have $\{e_\mu, if_0\}$, $\{f_0\}$, and $\{ie_k, ie_0, f_k, if_k\}$ as closed algebras. Even though $\mathscr{L}$ represents rotations in five-dimensional space-time, it turns out that its generators are almost isomorphic to those of the Poincaré group! This was a totally unexpected result. Translations are not considered here because the space is not uniformly curved and is time-dependent!

Notice that

$$g_{\mu\nu} \to g'_{\mu\nu} = \mathscr{L}^\wedge g_{\mu\nu} \mathscr{L} \tag{3.2}$$

under the transformation $b_\mu \to b'_\mu$. In four-space $\mathscr{L}\mathscr{L}^\ddagger = 1$ implies $\mathscr{L}^\ddagger\mathscr{L} = 1$. Since $g_{\mu\nu}$ is proportional to e_0, even in five-space, we get $\mathscr{L}^\wedge\mathscr{L}g_{\mu\nu}$ and must check if $\mathscr{L}^\wedge\mathscr{L} = 1e_0$ follows from $\mathscr{L}\mathscr{L}^\wedge = 1e_0$. The subgroups generated by (if_0) and (if_k) have the form

$$\cosh(\theta/2)(e_0) + \sinh(\theta/2)(if_0) \qquad \text{and} \qquad \cos(\theta/2)(e_0) + \sin(\theta/2)(if_k) \tag{3.3}$$

We readily check that $\mathscr{L}^\wedge\mathscr{L} = \mathscr{L}\mathscr{L}^\wedge = 1e_0$. Applying these subgroups to b_μ we find that (if_0) generates rotations in the (e_0), (if_0) plane. This is analogous to the Lorentz boost transformations along e_k. Also, (if_k) generates rotations in the (e_k), (if_0) plane. These appear to be ordinary rotations between the three macroscopic and one microscopic spatial dimensions. These 'physical' interpretations may be only figurative or approximations for small space-time regions where the curvature can be neglected. However, the interaction of two protons involves the curved space influence of each on the other when they pass. This cannot be strictly neglected and may be dominate at very high energies (close approach). It is probably better to think of $\mathscr{L}\mathscr{L}^{\wedge}$ covariance as a *formal* mathematical symmetry of the laws of nature. The rest frame of the galaxies, in the expanding universe, is the preferred frame in which to consider the laws of nature, including gravitation (the background radiation istropy frame). Lorentz symmetry need not be physically associated with the equivalence of reference frames.

To generate wave equations, we postulate

$$P = i\hbar b_\mu D^\mu \to P' = \mathscr{L}^\dagger P \mathscr{L} \tag{3.4}$$

as the differential operator in curved five-dimensional space. The simplest wave equation is

$$P\psi_a = 0 \to P'\psi'_a = 0 = (\mathscr{L}^\dagger P \mathscr{L})(\mathscr{L}^\wedge \psi_a) \tag{3.5}$$

This is the Dirac equation. To show this we go to a matrix representation

$$\left[i\hbar\,\partial^\mu \begin{pmatrix} \sigma_\mu & 0 \\ 0 & \sigma_\mu^\ddagger \end{pmatrix} + i\hbar\,\partial^4 \begin{pmatrix} 0 & -i\sigma_0 \\ i\sigma_0 & \end{pmatrix}\right] \begin{pmatrix} \psi_a \\ \phi_v \end{pmatrix} = 0, \qquad i\hbar\,\partial^4 \begin{pmatrix} \psi_a \\ \phi_v \end{pmatrix} \equiv -m \begin{pmatrix} \psi_a \\ \phi_v \end{pmatrix}$$

$$i\hbar\,\partial^\mu \sigma_\mu \psi_a - m(-i\phi_v) = 0, \qquad i\hbar\,\partial^\mu \sigma_\mu^\ddagger \phi_v - mi\psi_a = 0, \qquad \mu = 0, 1, 2, 3 \tag{3.6}$$

To finish the correspondence define $\psi_v \equiv -i\phi_v$, which gives

$$i\hbar\,\partial^\mu \sigma_\mu \psi_a = m\psi_v, \qquad i\hbar\,\partial^\mu \sigma_\mu^\ddagger \psi_v = m\psi_a, \qquad \mu = 0, 1, 2, 3 \tag{3.7}$$

Notice that we have postulated that ψ_a is an eigenstate of $i\hbar\,\partial^4$ with eigenvalue $-m$ to get the Dirac equation. The eigenvalue im gives the Tachyon equation. The eigenvalue zero gives the Weyl equation. This describes neutrinos. When $\partial^4\psi = 0$ we have χ^4 'constant'. Thus zero mass particles are '*stationary*' in position along the fifth dimension! Similarly, an electron at rest has $\partial^4\psi = -m\psi \neq 0$, so the particle is still '*moving*' along χ^4. Tachyons presumably are always 'moving' along χ^k and χ^4, if they exist. Looking at mass as an eigenvalue, rather than an *ad hoc* parameter, may eventually lead to an answer as to whether imaginary mass is possible in nature.

Here $P = P^\dagger = P_e^\dagger + P_f^\dagger = P_e^* + P_f^\ddagger = P$. Therefore $P^\wedge\psi_v = 0$ is not an independent equation, since $\psi_v = \psi_a^{\dagger\wedge}$ is always possible. If higher dimensions are physical, such that $P \neq P^\dagger$, then there are two basic spin-$\frac{1}{2}$ wave equations. The γ_5 matrix is (ie_0) in our notation. It commutes with e_μ but anticommutes with (if_0). This is why $m = 0$ allows ψ to be an eigenstate of (ie_0) (neutrino helicities). Notice also that $(if_0)(if_0) = (e_0)$, so its eigenvalues are ± 1. This means that it is the 'particle, antiparticle' operator; $(if_0)\psi = \pm 1\psi$, means that ψ is a state of only particles or only antiparticles. The generators (ie_3) and (if_0) commute and form the 'trunk' of the Lie algebra $\mathscr{L}\mathscr{L}^\wedge$. Thus spin and particle number (Lepton or Baryon) are the quantum numbers that come from the $\mathscr{L}\mathscr{L}^{\wedge}$ symmetry.

Next we consider second-order wave equations. The 5-vector inner product is

$$(P\,|A) \equiv \tfrac{1}{2}[(P^{\wedge}A) + (\;)^{\wedge}] \rightarrow (P'\,|A') = \mathscr{L}^{\wedge}(P\,|\,A)\mathscr{L} \qquad (3.8)$$

In flat space this gives

$$(P\,|\,P) = P^{\wedge}P = (P^0P^0 - P^kP^k - P^4P^4)e_0 = (P'\,|P') \qquad (3.9)$$

If we had used (f_0) instead of (if_0) for 5-vectors, we would get many problems here. Fortunately, the Dirac equation can be written in terms of either (f_0) or (if_0) and the form of equation (3.9) shows that (if_0) is preferable for forming 5-vectors. Since it is invariant under $\mathscr{L}\mathscr{L}^{\wedge}$, it may be useful to think of (f_0) as a sixth coordinate, related to time in the cosmic preferred frame, even though it does not enter the wave equations. If the gravity equation requires 16-vectors for space-time physics, then this idea would be directly useful. I like the idea of antimatter travelling backward with respect to (e_0) coordinate time but forward, like matter, for (f_0) cosmic time. This biases the universe toward matter over antimatter, which is how the world actually appears so far, experimentally.

The Klein–Gordon equation appears in equation (3.9) by $P^4\psi \equiv -m\psi$, so that $P^4P^4\psi = m^2\psi$. The Maxwell equation (Proca equation) has the form

$$(P\,|\,P)^{\dagger}A - P(P\,|\,A) = 0 \rightarrow$$

$$[\mathscr{L}^{\wedge}(P\,|\,P)\mathscr{L}]^{\dagger}[\mathscr{L}^{\dagger}A\mathscr{L}] - [\mathscr{L}^{\dagger}P\mathscr{L}]\,[\mathscr{L}^{\wedge}(P\,|\,A)\mathscr{L}] = 0 \qquad (3.10)$$

In five-space, but not for higher dimensions, we have $(P\,|\,P) \propto e_0$ and the $(P\,|\,P)^{\dagger}$ term can be replaced by $(P\,|\,P)$ because $(P\,|\,P)$ is invariant. Again and again we see that this number system has the machinery for some very general, and complicated, covariant physical laws. Let's hope 5-vectors are *enough* to describe relativistic physics. Notice that even here $(P\,|\,P)$ is complicated in curved space because $b_{\mu}{}^{\wedge}D^{\mu}b_{\nu}D^{\nu} \neq b_{\mu}{}^{\wedge}b_{\nu}D^{\mu}D^{\nu}$; but see Edmonds, 1975.

There is an important point to be made about mass and equation (3.10). It is *not* a free parameter to be stuck in the wave equation at will to generate other wave equations. I think this is where the efforts to find spin-$\frac{3}{2}$ and spin-2 covariant wave equations have fallen down. In equation (3.10), P is a 5-vector. Therefore, if it describes a particle with rest mass then $\partial^4\psi \neq 0$ and the mass term will appear 'automatically' in the equation! A similar statement applies to conserved currents. We have five dimensions (at least). In flat five-space $(P\,|\,P)$ and $(P\,|\,A)$ *can* be brought outside the inner product so that

$$(P\,|\,(P\,|\,P)^{\dagger}A) - (P\,|\,P(P\,|\,A)) = (P\,|\,J) = (P\,|\,P)^{\dagger}(P\,|\,A) - (P\,|\,P)(P\,|\,A) = 0 \qquad (3.11)$$

The second term in equation (3.10) is necessary (as is its minus sign) to get this result. I am very suspicious of the claim that $(P\,|\,A) \equiv 0$ is possible

(Lorentz gauge) as a 'subsidiary' condition to equation (3.10). Notice that J is in general a 5-vector! Maybe when you can neglect its fifth component, and space curvature, so that A has only four components, the gauge ideas are valid.

4. *Wave Equation Couplings*

The usual Dirac current, $\bar{\psi}\gamma_\mu\psi$, looks harmless and simple enough. However, in our earlier efforts with (f_0) type 5-vectors and 8-vectors we found that it was a real problem. Here it is much simpler because $g_{\mu\nu}$ transforms simply under $\mathscr{L}\mathscr{L}^\wedge$ transformations. The Dirac current is simple because it is incomplete. It neglects the requirements of Einstein covariance!

We introduce another inner product for the hypercomplex numbers

$$\langle A \mid B \rangle \equiv \tfrac{1}{2}[(A^\dagger B) + (\ )^\wedge] \tag{4.1}$$

Then

$$\langle \psi_a \mid b_\mu \psi_a \rangle = \tfrac{1}{2}[(\psi_a^\dagger b_\mu \psi_a) + (\ )^\wedge] = \langle \mid \rangle^\dagger, \qquad b_\mu^\dagger = b_\mu \tag{4.2}$$

Here $\{b_\mu\}$ is $\{e_\mu, if_0\}$. Since $\langle | \rangle = \langle | \rangle^\dagger$ it follows that it has no (f_k), (ie_μ) or (if_k) components. Since $\langle | \rangle = \langle | \rangle^\wedge$ it follows that it has no (e_k), or (if_0) components. Therefore $\langle | \rangle = \alpha e_0 + \beta f_0$. To remove the f_0 component, we add $\langle | \rangle^{\wedge\vee}$ which acts as follows $(e + f)^{\vee\wedge} = (e - f)$. Putting this all together

$$\tfrac{1}{2}[\langle \psi_a \mid b_\mu \psi_a \rangle + \langle \psi_a \mid b_\mu \psi_a \rangle^{\vee\wedge}] \tag{4.3}$$

is proportional to e_0, and therefore commutes with $\mathscr{L}$, which was our objective. To obtain Einstein covariance, we must introduce b^μ and sum on $\mu = 0, 1, 2, 3, 4$. Now we check the $\mathscr{L}\mathscr{L}^\wedge$ covariance. Our basic postulate is that $b_\mu \to \mathscr{L}^\dagger b_\mu \mathscr{L}$. Now we define

$$b^\mu \equiv b_\mu g^{\mu\nu}, \qquad g_{\lambda\nu} g^{\mu\lambda} \equiv \delta_\nu{}^\mu \tag{4.4}$$

Because $g_{\mu\nu} \to \mathscr{L}^\wedge g_{\mu\nu} \mathscr{L} = \mathscr{L}^\wedge \mathscr{L} g_{\mu\nu} = g_{\mu\nu}$ we can easily compute $g^{\mu\nu}$ which is also $\mathscr{L}\mathscr{L}^\wedge$ invariant for 5-vectors. These nice properties of $g_{\mu\nu}$ are lost for higher dimensions. This probably means there are no higher dimensions or that the Dirac current is not physically meaningful. Notice that $\psi_a^{\dagger\wedge}\psi_a{}^\wedge$ gives a 'current' which transforms as $\mathscr{L}^\dagger(\ )\mathscr{L}$ for ψ_a transforming as $\mathscr{L}^\wedge\psi_a$. This is also true of

$$\tfrac{1}{2}[\langle \psi_a \mid b_\mu \psi_a \rangle + \langle \psi_a \mid b_\mu \psi_a \rangle^{\wedge\vee}]\, b_\nu g^{\mu\nu} \tag{4.5}$$

which is quite a 'monstrosity' of a coupling to the A field. Actually

$$\psi_a^\dagger b_\mu \psi_a \to (\mathscr{L}^\wedge \psi_a)^\dagger (\mathscr{L}^\dagger b_\mu \mathscr{L})(\mathscr{L}^\wedge \psi_a) = \psi_a^\dagger \mathscr{L}^{\wedge\dagger} \mathscr{L}^\dagger b_\mu \mathscr{L} \mathscr{L}^\wedge \psi_a = \psi_a^\dagger b_\mu \psi_a \tag{4.6}$$

i.e. it is invariant under $\mathscr{L}\mathscr{L}^{\wedge}$. We see that Einstein covariance can be important for determining couplings. Consider $\Psi_v^\dagger b_\mu^\wedge \Psi_v$ where $\Psi_v \to \Psi_v' = \mathscr{L}^\dagger \Psi_v \mathscr{L}$. Then $\Psi_v^\dagger b_\mu^\wedge \Psi_v \to \mathscr{L}^\dagger(\ )\mathscr{L}$ under $\mathscr{L}\mathscr{L}^\wedge$ but is not a satisfactory current because it needs b^μ to make it Einstein covariant.

Couplings like $\bar{\psi}\gamma_\mu(1+\gamma_5)\psi$ have a similar problem. In our notation this becomes something like

$$\psi_a^\dagger f_0 f_\mu(e_0 + ie_0)\psi_a = \psi_a^\dagger e_\mu(e_0 + ie_0)\psi_a$$
$$= \psi_a^\dagger(e_\mu + ie_\mu)\psi_a = \psi_a^\dagger e_\mu \psi_a + \psi_a^\dagger(ie_\mu)\psi_a \tag{4.7}$$

We would generalize this to 5-vectors in the form $\psi_a^\dagger b_\mu \psi_a + \psi_a^\dagger(?)\psi_a$. Does $ie_\mu \to \{ie_\mu, f_0\}$? Then under $\mathscr{L}\mathscr{L}^\wedge$, how does ie_μ transform? Like e_μ? If we have 16-vectors, then $e_\mu + ie_\mu$ would generalize in a natural way to $\{e_\mu + ie_\mu, f_\mu + if_\mu\}$. But then again we have the problem with satisfying Einstein covariance.

If this 5-vector formulation is correct, physically, it will show up by leading to the correct kinds of relativistic couplings and 'higher spin' wave equations. It gives us new eyes with which to look at the old problems of particle theory which have resisted solution for decades. Of the many groping variations of this theme that I have generated these past several years, this 5-vector structure looks most elegant, *simple* and promising. The next step is 'second quantization' and hence experimental prediction. Since mass enters as an observable eigenvalue of the coupled field system, the renormalization question may take a very new turn, as did Lorentz covariance. Einstein's 'general covariance' is really replaced by the postulate—sum on *all* indices. This requires modification of Einstein's gravity law, which will test this postulate.

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